MATH 6021 Lecture 12 11/30/2020

Recall: Min-Max Theory (Almgren-Pitts, Simon-Smith, Guaraco)

Yau Conj (1982): $\exists$ ∞'ly many min. hypersurf. in ALL (M^{n+1}, g)

for "general" metrics

- Thm A: (Marques-Neves) YES for $Ric_g > 0$ or "Frankel Property" holds
- Thm B: (Song) YES in general.
- Thm C: (Liokumovich-Marques-Neves) $\{\omega_p\}$ satisfy a Weyl Law.

For "generic" metrics,

Thm D: (Irie-Marques-Neves) Min. hypersurf. are "dense".

Thm E: (Marques-Neves-Song) Min. hypersurf. are "equi-distributed".

Q: Is there a "Morse theory" for the Area Functional?

Morse theory: M manifold, $f: M \to \mathbb{R}$ "Morse" $\rightsquigarrow$ index of crit. pts $\rightsquigarrow$ reconstruct M topologically

~ which handle?

Given M^{n+1} manifold
g : Riem metric on M
$\Rightarrow$ $\mathcal{A}_g : \mathcal{Z}_n(M;\mathbb{Z}_2) \to \mathbb{R}$

$\xRightarrow{\text{B. White}}$ Bumy Metric Thm: $\mathcal{A}_g$ is "Morse" for generic g

Q: Control the index of the "crit pts" / min hypersurf?

Morse Index Conjecture (Marques-Neves)

For generic (bumpy) (M^{n+1}, g), $\exists$ seq. $\{\Sigma_k\}_{k\in\mathbb{N}}$ of min. hypersurfaces in M

s.t. (1) $\text{index}(\Sigma_k) = k$. [Recall: $\mathcal{Z}_n(M;\mathbb{Z}_2) \sim \mathbb{RP}^\infty$]

(2) $C^{-1} k^{\frac{1}{n+1}} \leq \text{Area}(\Sigma_k) \leq C k^{\frac{1}{n+1}}$ for some $C > 0$.

The proof consists of 3 components:

(I) <u>Existence</u>: do multi-parameter min-max (Almgren, Marques-Neves)

$\rightsquigarrow$ stationary varifold V_k (obtained via varifold limit)

<u>Main Difficulty</u>: convergence is weak & multiplicity issue

(II) <u>Morse Index characterization</u> (Marques-Neves '16)

[Heuristic: k-parameter sweepout $\Rightarrow$ index$(V_k) \overset{=?}{\leq} k$]

Assume "multiplicity one", then index$(V_k) = k$

(III) <u>Multiplicity One Conj</u>: "multiplicity one" holds for generic (M^{n+1}, g).

Solved by X. Zhou 2020, based on earlier work of

Zhou-Zhu '19 on min-max theory for prescribed mean curvature

<u>Q</u>: Can we control the geometry/topology of $\{\Sigma_k\}$?

(Partial results in $\dim(M) = 3$: Chodosh-Mantoulidis 2020)

Two applications of min-max theory

(Geometry): <u>Willmore Conjecture</u> (1965)

$\Sigma \subset \mathbb{R}^3$ closed embedded surface of genus ≥ 1

$\Rightarrow$ The "Willmore energy"

$$\frac{1}{4}\int_\Sigma H^2 da \geq 2\pi^2$$

Moreover, "=" holds iff

$\Sigma \cong$ "Clifford torus".

<u>Marques-Neves '14</u>: Willmore Conj. holds.

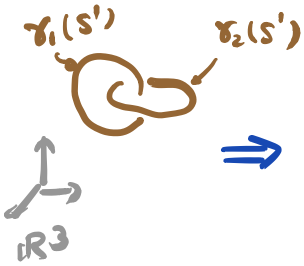
(Topology) Freedman-He-Wang Conj. (1994)

$\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$

"non-trivial" link, ie

$\gamma_1(S^1) \cap \gamma_2(S^1) = \phi$

and "linking number" = 1



$\Rightarrow$ The Möbius energy

$$\iint_{S^1 \times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2} \, ds\,dt \geq 2\pi^2$$

Moreover, "=" holds iff the link is a standard "Hopf link".

Marque-Neves ('14?) F-H-W. Conj holds.

Remark: Both "energy" are conformally invariant.

We will describe Willmore Conj. and its proof in more detail.

Willmore Conjecture

$\Sigma^2 \subseteq \mathbb{R}^3$ cpt surface $\rightsquigarrow$ $\mathcal{W}(\Sigma) := \frac{1}{4} \int_\Sigma H^2 da$ "Willmore energy"

Remarks: • $\mathcal{W}$ is conf. invariant, ie. $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a conformal diffeo.

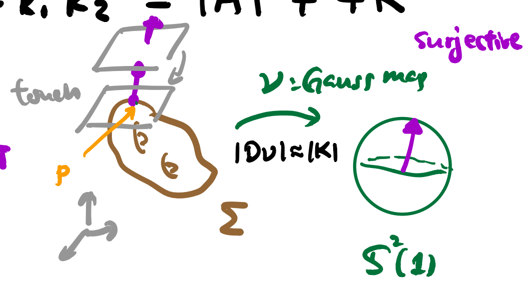
$$\Rightarrow \quad \mathcal{W}(\varphi(\Sigma)) = \mathcal{W}(\Sigma)$$

• $\mathcal{W}(\text{round sphere}) = \mathcal{W}(S^2(1)) = \frac{1}{4} \int_{S^2(1)} 2^2 da = \text{Area}(S^2(1)) = 4\pi$.

Thm (Willmore '65) $\mathcal{W}(\Sigma) \geq 4\pi$ & "=" holds $\Leftrightarrow$ $\Sigma \cong$ round sphere.

"Geometric Proof": $H^2 = (\kappa_1 + \kappa_2)^2 = (\kappa_1 - \kappa_2)^2 + 4\kappa_1\kappa_2 = \overbrace{|\mathring{A}|^2}^{\geq 0} + 4K$

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma H^2 \geq \frac{1}{\cancel{4}} \int_{\Sigma \cap \{K \geq 0\}} \cancel{4} K \geq \text{Area}(S^2(1)) = 4\pi$$



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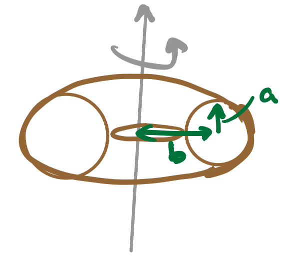
Q: What about the "next smallest" energy?

Conj: $\mathcal{W}(\Sigma) \geq 2\pi^2$ if genus$(\Sigma) \geq 1$

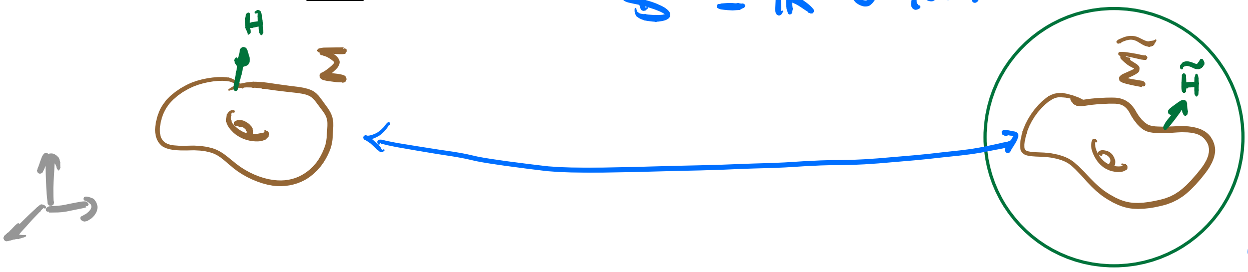


Note: Willmore checked rotationally symmetric tori

We can reformulate the question to surfaces in $\mathbb{S}^3$.

$$\mathbb{R}^3 \xleftrightarrow{\text{stereographic projection}} \mathbb{S}^3$$

$\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$



curvature of $\mathbb{S}^3$

$$\mathcal{W}(\Sigma) := \frac{1}{4}\int_\Sigma H^2 da \quad = \quad \mathcal{W}(\tilde\Sigma) = \int_{\tilde\Sigma} \left(1 + \frac{1}{4}\tilde{H}^2\right) da$$

Key Observation: minimizes $\mathcal{W}$ in $\mathbb{R}^3$ $\approx$ minimizes "area" in $\mathbb{S}^3$

Willmore surfaces in $\mathbb{R}^3$ $\longleftrightarrow$ min. surface in $\mathbb{S}^3$

Examples:

$\mathbb{R}^3$, $\mathcal{W} = \frac{1}{4}\int H^2$		$\mathbb{S}^3$, $\mathcal{W} = \int 1 + \frac{1}{4}\tilde{H}^2$	
round spheres	$\mathcal{W} = 4\pi$	tot. geodesic $\mathbb{S}^2$,	Area $= 4\pi$
Willmore's torus	$\mathcal{W} = 2\pi^2$	"Clifford torus".	Area $= 2\pi^2$.
		$S^1(\frac{1}{\sqrt2}) \times S^1(\frac{1}{\sqrt2}) \subseteq \mathbb{S}^3 \subseteq \mathbb{R}^4 = \mathbb{C}\times\mathbb{C}$	

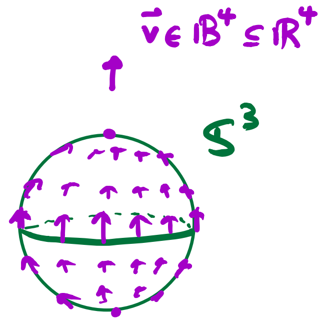
Partial Results:

- Li-Yau '82: Σ immersed $\Rightarrow$ $\mathcal{W}(\Sigma) \geq 8\pi$ $(> 2\pi^2)$.
- L. Simon '93: existence of $\mathcal{W}$-minimizing torus
- Ros '99: Conj. holds under the assumption of antipodal symmetry.

Finally, Marques-Neves '14 answered Willmore Conj. affirmatively.

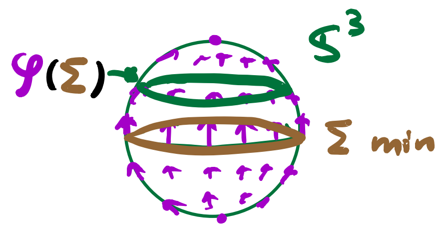
"Sketch of Marques-Neves' Proof"

Recall some preliminary results:



(i) $\mathrm{Conf}(\mathbb{S}^3) := \{ \varphi : \mathbb{S}^3 \to \mathbb{S}^3 : \text{conf. diffeo.} \} \cong \mathbb{B}^4$

(ii) (Montiel-Ros)



For any min. surf. $\Sigma \subseteq \mathbb{S}^3$, then

$$\mathrm{Area}(\Sigma) \geq \mathrm{Area}(\varphi(\Sigma)) \quad \forall \varphi \in \mathrm{Conf}(\mathbb{S}^3)$$

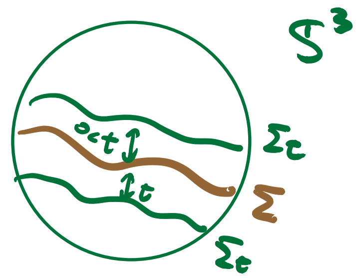
(Urbano) For any min surf. $\Sigma \subseteq \mathbb{S}^3$, $\Sigma \neq \mathbb{S}^2$, then

$$\mathrm{index}(\Sigma) \geq 5 \quad \& \quad \text{"=" holds} \iff \Sigma = \text{Clifford torus}$$

(iii) (Ros) For any surf. (not nec. min.) $\Sigma \subseteq \mathbb{S}^3$, let $\Sigma_t := \{ x \in \mathbb{S}^3 \mid \mathrm{dist}_{\mathbb{S}^3}(x, \Sigma) = t \}$ (signed)

then $\mathrm{Area}(\Sigma_t) \leq W(\Sigma)$

& if $\Sigma \neq \mathbb{S}^2$, then "=" holds $\iff t = 0$, Σ min.



We now go into the proof of Willmore Conj.

<u>$2\pi^2$ Theorem</u>: If $\Phi : I^5 = [0,1]^5 \xrightarrow{\text{cts}} \mathcal{Z}_2(\mathbb{S}^3 ; \mathbb{Z})$ s.t.

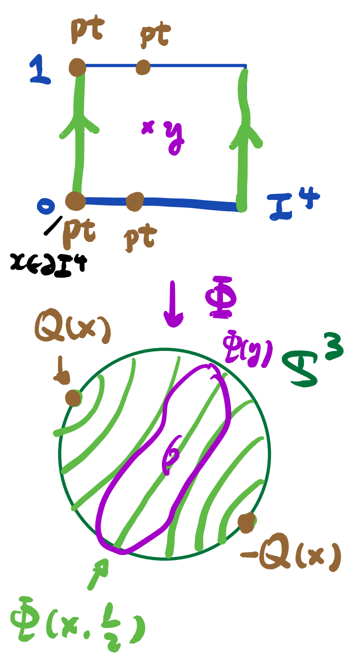
(1) $\Phi(x, 0) = \Phi(x, 1) = pt$ for any $x \in I^4$

(2) For any $x \in \partial I^4$,

$$\{\Phi(x,t)\}_{t \in [0,1]} = \text{"standard" sweepout of } \mathbb{S}^3 \text{ by round spheres centered at } Q(x) \in \mathbb{S}^3$$

(3) $\Phi(x, \frac{1}{2}) = \partial B_{\pi/2}(Q(x))$

(4) $Q : \partial I^4 \approx \mathbb{S}^3 \xrightarrow{\text{cts}} \mathbb{S}^3$ has $\deg Q \neq 0$.



THEN, $\exists\, y \in I^5$ s.t. $\mathrm{Area}(\Phi(y)) \geq 2\pi^2$.

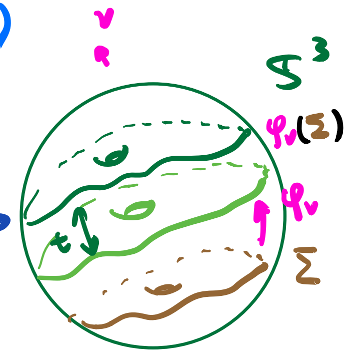
Assume this at the moment, we prove Willmore Conjecture.

Given any closed embedded surface $\Sigma \subseteq S^3$, genus$(\Sigma) \geq 1$.

We can construct a 5-parameter "canonical family"

$$\Phi : \underbrace{\overset{\text{Conf}(S^3)}{B^4} \times \overset{\text{dist}_{S^3}}{(-\pi,\pi)}}_{\approx I^5} \longrightarrow \mathcal{Z}_2(S^3;\mathbb{Z})$$

$$\Phi(v,t) := \{x \in S^3 \mid \text{dist}_{S^3}(x, \varphi_v(\Sigma)) = t\}$$



One can show that Φ satisfies (1)-(3) by a cts extension to ∂I^5. Also, one can prove * $\deg Q = \text{genus}(\Sigma) \geq 1$ *

so (4) is also satisfied.

$2\pi^2$ Thm $\Rightarrow \exists\, \overset{(v,t)}{y} \in I^5$ s.t. $\mathrm{Area}(\Phi(y)) \geq 2\pi^2$.

Recall by Prelim. result (iii) + conf. invariance of $\mathcal{W}$.

$$\mathcal{W}(\Sigma) = \mathcal{W}(\varphi_v(\Sigma)) \geq \mathrm{Area}(\Phi(y)) \geq 2\pi^2.$$

$\square$

Idea of Proof for "$2\pi^2$ Thm":

Given Φ as in the theorem, consider 5-parameter min-max:

$$L([\Phi]) = \inf_{\Phi' \sim \Phi} \left(\sup_{x \in I^5} \mathrm{Area}(\Phi'(x)) \right)$$

Min-Max Theory $\Rightarrow$ $L([\Phi])$ is achieved by the area of some min. surf. (up to multiplicity).

ie $L([\Phi]) = m_1 \operatorname{Area}(\Sigma_1) + m_2 \operatorname{Area}(\Sigma_2) + \cdots + m_q \operatorname{Area}(\Sigma_q)$

<u>Note</u>: $\operatorname{Area}(\Sigma_i) \geq 4\pi \Rightarrow q = 1$ & $m_1 = 1$ $(\because 8\pi > 2\pi^2)$

so, $L([\Phi]) = \operatorname{Area}(\Sigma)$

Marques-Neves' Morse index upper bound,

k-parameter sweepouts $\xRightarrow{\text{min-max}}$ min. surf of index $\leq k$

Now, Φ is 5-parameter family $\Rightarrow$ $\operatorname{index}(\Sigma) \leq 5$

Urbano's result $\Rightarrow$ $\operatorname{index}(\Sigma) = 5$ and $\Sigma \cong$ Clifford torus

Area $= 2\pi^2$

$\square$